Digital Signal Processing and Fourier Transforms: Everyday Image Editing

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In this paper, the methods of digital signal processing, specifically image processing, will be analyzed. An image or audio can be represented as a signal in a specific domain, which is fed into a system, and delivered as an output in another domain of interest. Signals can be represented by the time domain, the frequency domain, the spatial domain, etc. The Fourier transforms are used to transform between different domains. In this paper, the Fourier transformation of an image from the spatial domain to the frequency domain will be analyzed and discussed.

Keyword: Signal Processing, Image Processing, Fourier Transform

I. INTRODUCTION

Digital Signal processing has many important applications in real life, including audio signal processing, digital image processing, speech recognition, etc. Audio signal processing is used for active noise control in earphone, for generating electronic audio signals in music, and to change how an audio signal sounds in audio effects. Image processing is used for pattern recognition, mapping the moon's surface, imaging the human body to detect breast diseases [3] or imaging the brain (tomography) in medical imaging.

Digital Signal Processing (DSP) takes in a signal such as sound, temperature, or position, and analyze or convert it to produce an output. For example, in image processing, the input signal is an image in the spatial domain (i.e. variables in terms of distances and positions), and the output signal is in the frequency/Fourier domain (i.e. variables are spatial frequencies). All signals can be decomposed into a weighted sum of sines and cosines, which is related to Fourier transform. Fourier transform and Laplace transform can be used to analyze and convert the original signal from one domain to another.

The original signal of an image is in the spatial domain, where the image is represented in terms of the spatial coordinates of its pixels/components. In image processing tools, a technique that is often used is sharpening or smoothing the features of an image. However, image signals in the spatial domain are not useful, as one cannot change the sharpness of an image by moving elements/pixels of an image around. Now, a

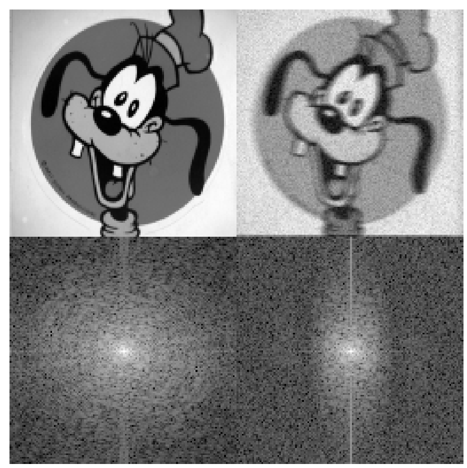

FIG. 1. The top left is the original image of a goofy in the spatial domain, and the top left is a smoothed goofy. The bottom images are frequency domain images that corresponds to the spatial images on the top. The original goofy is smoothed only in the horizontal direction, so the frequency in the horizontal direction of the bottom right image is being attenuated.[1]

sharp edge means a discrete change in the structure of an image. Hence, the goal is to use a domain that represents the rate of change of the structure/pixels of an image, which is the (spatial) frequency domain. The new image is a weighted sum of sine and cosine wave in the frequency domain, and each point in this image represents a particular frequency in the original spatial domain image. An example of smoothing an image is shown in Fig. 1 Sharpening or smoothing the structure of images comes down to changing the coefficients of certain sine/cosine waves or setting them to zero.

II. MATERIALS AND METHODS

The goal is to transform from the spatial domain to the frequency domain. This can be done using Fourier transform. Sine and cosine waves of the same frequency ω can be represented as a complex exponential by Euler's formula:

$$e^{i\omega} = cos(\omega) + isin(\omega) \tag{1}$$

Since images are a superposition of sine and cosine waves, all images in the spatial domain can be represented as a sum of complex exponentials of the frequency domain [2]. For simplicity, let's first consider the transformation of two spatial variables, as most images in real life are two dimensional. All two dimensional images can be represented as a function $f(x, y)$ of two spatial variables x and y [5]. Let ω_1 and ω_2 be frequency variables, with units of radian/sample. The frequency-domain representation of $f(x, y)$ is then $F(\omega_1, \omega_2)$. Fourier transform is used to transform from $f(x, y)$ to the frequency-domain representation $F(\omega_1, \omega_2)$ by the series expansion [2]:

$$F(\omega_1, \omega_2) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y) e^{-i\omega_1 x} e^{-i\omega_2 y} \tag{2}$$

Note when one shrink the interval between each (x,y) data point to an infinitesimal scale

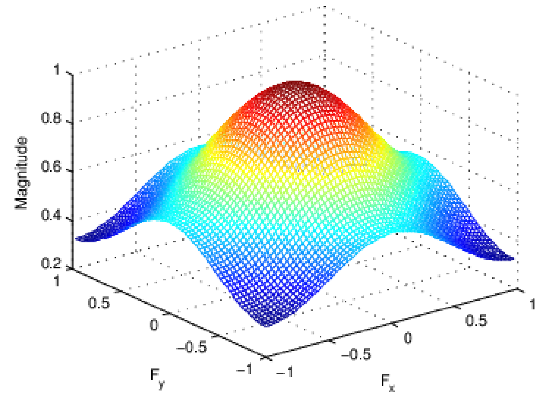

FIG. 2. Gaussian function representing a smoothed point in the frequency domain. F_x and F_y represent the frequency coordinates in the image. Note, unlike a delta function which would look like an approximately vertical line, the Gaussian is more spread out around the point.

(i.e. $\Delta x, \Delta y \to 0$), the summation becomes an integral. This means that an approximately infinite number of (x,y) data points are taken in the spatial domain of the image, which is not easy to achieve in real life applications. Usually, the integral form of Eq. 2 is used when one can find a continuous function to describe the image in the frequency domain.

After transforming the image into the frequency domain, one can perform all types of operations on it to alter the final image. One example is image smoothing. Each point in the frequency domain image can be viewed as a delta function (a spike at that point) with some particular coefficient. The image is then a sum of scaled and shifted delta functions [6]. One can smooth the image out by spreading the delta function, which would result in a Gaussian function. The smooth result is known as a Gaussian blur. An example of a frequency point represented by the Gaussian in shown in Fig.2. Transforming a point into the pillbox function is also a way to smooth the image. There are also techniques to sharpen an image, such as transforming a point into a sinc function.[6]

After altering coefficients of certain exponentials/performing some operations on the Fourier image, one can transform the image signal back to the spatial domain by transforming the new function $F'(\omega_1, \omega_2)$ (edited image in the frequency domain) to $f'(x, y)$ by [2]:

$$f'(x,y) = \frac{1}{4\pi^2} \int_{\omega_1=-\pi}^{\pi} \int_{\omega_2=-\pi}^{\pi} F'(\omega_1, \omega_2)\, e^{i\omega_1 x} e^{i\omega_2 y} d\omega_1 d\omega_2 \tag{3}$$

Note the equation here is an integral as when altering coefficients in the frequency domain, it is usually easier to find a function to describe the image in the frequency domain. In general, in both cases image editing in real life uses the discrete form of the transformation. The new function f'(x,y) describes the final edited image. Examples of images of shapes in the spatial domain being transformed into the frequency domain are shown in Fig. 3.

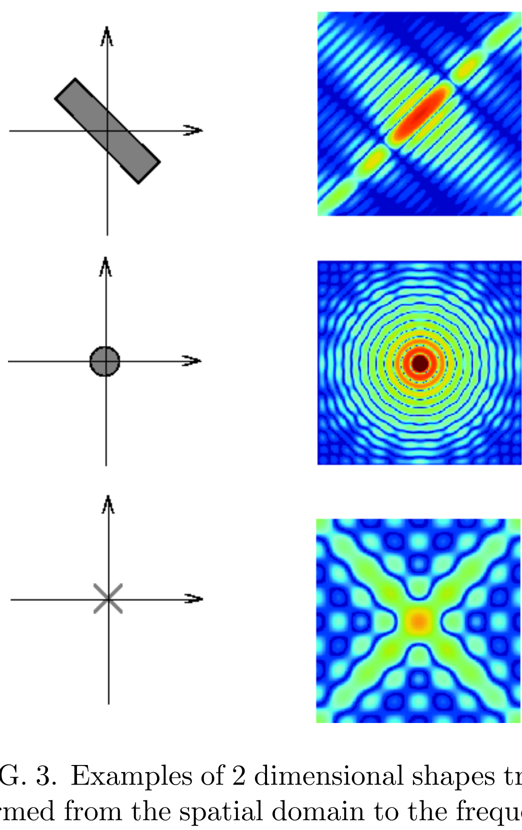

FIG. 3. Examples of 2 dimensional shapes transformed from the spatial domain to the frequency domain [2]

III. RESULTS AND ANALYSIS

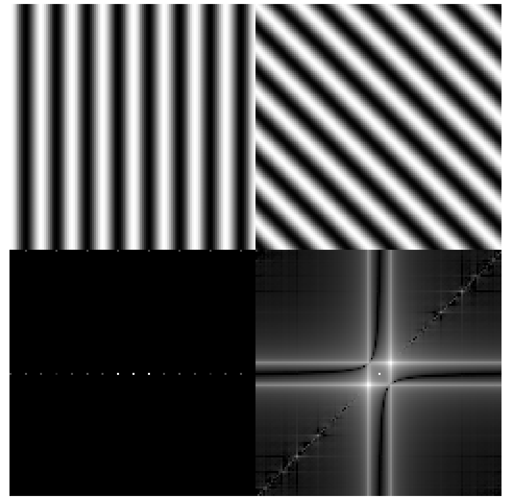

FIG. 4. The top two images are in the spatial domain, and the bottom two are the corresponding images in the frequency domain.

The sharpness of images seen in everyday life can be modified by transforming from the spatial domain to the frequency domain, changing the frequencies, and then transforming back to the spatial domain to produce a smoother or sharper image. From this, Fourier transform is a useful tool to transforming images between different domains in order to change certain characteristics of the images, such as that of the Goof in Fig. 1.

Since images are represented as cosine and sine functions, images that are only composed of sine or cosine waves have simple Fourier transforms, and simpler images in the frequency domain, as shown in the left two images of Fig. 4. The frequency domain image (bottom one) is just a simple image of three white dots. However, if the image is rotated by an angle of 90 degrees, the frequency domain image becomes more complicated (as shown in lower image of the right column) because the spatial domain image is no longer a simple sine or cosine wave, but periodic constructive and destructive interference of waves to form diagonal patterns. There are also other such images that are used to represent certain combinations of sines and cosines; in other words, other images that

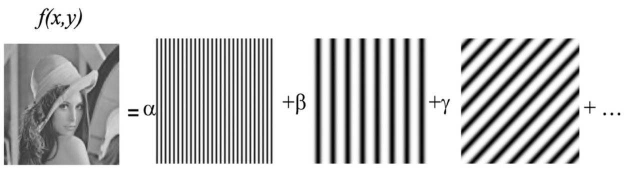

FIG. 5. An image can be decomposed into a weighted sum of bases. The images on the right can be viewed as different bases in the spatial domain. α, β, γ are coefficients of the exponential corresponding to each basis; they represent how much weight each basis is assigned to in the final image $f(x, y)$ [1].

represent different exponentials. These images are usually used bases and summed up to represent more complicated images. As shown in Fig. 5, one can decompose an image into a sum of these bases.

IV. DISCUSSION

With Fourier transforms, one can easily edit certain characteristics of an image, such as relative sharpness/blurriness at different locations of, relative brightness, and the contrast between patterns of an image. Other mathematical methods have also been used in image processing, such as convolution and Laplace transform. Convolution is the sum of the multiplication of two sets of signals to produce another set of numbers [4]. Usually the input two sets are matrices analogous to the spatial domain, and the output set is of interest for editing the image. The method of the Gaussian as discussed in Section II is used in convolution. Laplace transform is similar to Fourier transform, but it is more widely used in other fields of signal processing, such as speech recognition and control systems. Applications of Laplace transform in control systems include controlling aircraft systems and their stability; these are further topics for investigation.

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